Design of Trellis Codes for Source Coding with Side Information at the Decoder

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Abstract

The problem of source coding with side information at the decoder arises in many practical scenarios. Although this problem has been well characterized in information theory, particularly by the work of Wyner and Ziv, there is still lack of successful algorithms for it. In this paper, we use trellis codes to approach the theoretical limit. An embedded trellis code structure is proposed, and its properties are examined. Using this structure, we can achieve the granular gain at the encoder as well as the coding gain at the decoder. Simulation results show that the proposed scheme outperforms the algorithms reported in the literature. It is also indicated that the performance of the proposed algorithm can approach the information-theoretic limit at high rate as the trellis complexity increases.

1 Introduction

Consider the source coding problem depicted in Fig. 1. An information source \( X \) is encoded at the encoder and a description \( \hat{X} \) is reproduced at the decoder with distortion

\[
D = E[d(X, \hat{X})],
\]

(1)

where \( d(\cdot, \cdot) \) is some distortion measure. The difference between the problem shown in Fig. 1 and the conventional rate-distortion problem is that, the decoder has access to a side information \( Y \), which is correlated to the source \( X \) with some joint probability density function (pdf) \( f(x, y) \). With this side information, one can anticipate that an encoder output \( Z \) of a lower rate denoted by \( R_e \), compared to the rate-distortion function \( R(D) \) of the source \( X \), is sufficient for the decoder to reproduce \( \hat{X} \) within the prescribed distortion \( D \). The objective is to encode \( X \) most efficiently.

This problem arises in many practical scenarios. For example, in a local area, weather or seismic data is collected at several distributed observation stations (called satellite stations) and is to be transmitted to a central station for further processing. The central station also collects the data by itself. If these stations are not very far
Figure 1: Source coding with side information at the decoder.

apart, the data observed at the satellite stations would be highly correlated to that observed at the central station. With the correlated side information at the central station, the satellite stations can probably transmit their own data at reduced rates, which in turn reduces the requirements of transmission bandwidth and/or power. Other application scenarios include distributed database, stereo and multi-camera systems, enhancement of analog TV by using a digital channel to transmit coded information [1], and reducing retransmission in time-varying channels [2], et al.

The information-theoretic aspects of this problem have been studied by Wyner and Ziv [3, 4]. It has been shown that, the rate-distortion function of $X$ given that the decoder observes $Y$ is

$$
\hat{R}(D) = \inf_{Z: Y \rightarrow X \rightarrow Z, I(Z;X,Y) \leq D} I(X;Z|Y),
$$

(2)

where $Y \rightarrow X \rightarrow Z$ denotes that $Y$, $X$ and $Z$ form a Markov chain. Note that

$$
\hat{R}(D) \geq R_{XIY}(D),
$$

(3)

where $R_{XIY}(D)$ is the rate-distortion function when the side information $Y$ is also available at the encoder. Although the equality in (3) does not hold in general, information-theoretic result shows [4] that if $X$ and $Y$ are jointly Gaussian and squared error distortion measure is considered, the encoder without knowledge of the side information can do as well as those which have that information. In fact, suppose $X \sim N(0, \sigma_X^2), Y = X + U$ with $U \sim N(0, \sigma_U^2)$ and independent of $X$, we have

$$
\hat{R}(D) = R_{XIY}(D) = \frac{1}{2} \log \frac{\sigma_X^2 \sigma_U^2}{D(\sigma_X^2 + \sigma_U^2)} = R(D) - C(\frac{\sigma_X^2}{\sigma_U^2}),
$$

(4)

where $C(\sigma_X^2/\sigma_U^2)$ is the capacity of the additive white Gaussian noise (AWGN) channel with signal-to-noise ratio (SNR) $\sigma_X^2/\sigma_U^2$.

It is instructive to see how (2) is achieved theoretically, which will serve as the guidelines for the design of practical algorithms. The proof of achievability of (2) relies on the random coding and random binning arguments [3]. A quantization codebook is first randomly generated, and then the codewords are randomly assigned to a smaller number of bins. At the encoder, the source $X$ is quantized using the codebook, but instead of sending the index of the resulting codeword, the index of the bin which contains that codeword is sent to the decoder. If the number of codewords in each bin is small enough, with high probability one and only one codeword in each
bin will be jointly typical with the side information $Y$. The decoder, which observes both $Y$ and the bin index, can thus reproduce the codeword with high probability.

The design of practical algorithms for this problem follows from the above idea. Nevertheless, the quantization codebook and the binning process need to be designed constructively. Previous work has been reported in [5] and [6]. In [5], scalar quantization codebook is employed while trellis coset decomposition is applied to the binning process. The work in [6] uses lattice quantization codebook and the lattice code is divided into similar sublattices during the binning process.

However, there are still big gaps between the performances of these algorithms and the information-theoretic limit. The goal of this work is to push the performance of the practical algorithm close to the theoretical limit under the same quantization-binning framework. For example, in the Gaussian case (c.f. (4)), we require the performance of the quantizer to approach $R(D)$, and that of the binning process to approach the capacity $C(\sigma^2_Y/\sigma^2_0)$ simultaneously. In particular, we choose to use trellis codes for both processes in order to obtain the benefits of high dimensional coding.

To that end, we propose an embedded trellis code structure, so that both the overall trellis code and the subcode in each bin are good trellis codes. The properties of the proposed code structure, such as non-catastrophe and minimality, are studied, since they determine the trellis complexity and error propagation. We then show that this trellis structure can achieve both the granular gain for quantization and the coding gain for recovering the quantization index at the decoder (which is equivalent to channel decoding with the side information as the channel output). Simulation results show that, with similar complexity at the decoder, the proposed structure attains 1 dB more coding gain than the algorithm in [5] in addition to the targeted granular gain which is greater than 1 dB. Simple comparison is also made with the work in [6]. The gap between the performance of the proposed algorithm and the theoretical limit is analyzed at later sections of this paper. It is shown that, with a high-rate assumption, the theoretical limit can indeed be approached by the proposed algorithm.

2 Embedded trellis codes

Good trellis codes have been designed for quantization [7] and modulation [8], and performance close to the rate-distortion function and performance close to the channel capacity have been achieved respectively. However, in general, these codes are not suitable for the problem of source coding with side information at the decoder. For example, a trellis code optimized for quantization may not guarantee the performance of resolving the quantization index in its subcode at the decoder; while trellis codes optimized for the subcodes may not add up to a good trellis-coded quantizer. Hence, our design should consider the performance of the overall trellis code and that of each subcode simultaneously. In this section, we present an embedded trellis code structure and study its properties. Its importance for this problem will be made clear in the next section.

Consider one dimensional trellis codes as those used in [7], and Ungerboeck's set
partitioning principle [8] is used. Fig. 2 gives an example of the signal set and a 4-state trellis.

\[
\begin{array}{cccc}
D_0 & D_1 & D_2 & D_3 \\
\rightarrow & \uparrow & \rightarrow & \uparrow \\
0/D_0 & 1/D_1 & 0/D_2 & 1/D_3 \\
\rightarrow & \uparrow & \rightarrow & \uparrow \\
0/D_0 & 1/D_1 & 0/D_2 & 1/D_3 \\
\rightarrow & \uparrow & \rightarrow & \uparrow \\
0/D_0 & 1/D_1 & 0/D_2 & 1/D_3 \\
\rightarrow & \uparrow & \rightarrow & \uparrow \\
0/D_0 & 1/D_1 & 0/D_2 & 1/D_3 \\
\end{array}
\]

Figure 2: Example of signal set and trellis.

Denote the generator matrices of two such trellis codes (rate-\(\frac{1}{2}\) convolutional codes) as

\[
G_i(D) = \begin{pmatrix} g_{i1}(D) \\ g_{i2}(D) \end{pmatrix}, \quad i = 1, 2. \tag{5}
\]

We construct the generator matrix of our embedded trellis code as

\[
G(D) = \begin{pmatrix} g_{11}(D) & 0 \\ g_{12}(D) & g_{21}(D) \\ 0 & g_{22}(D) \end{pmatrix}. \tag{6}
\]

Then, the new code is of rate-\(\frac{2}{3}\), and can be written as

\[
c(D) = G(D)a(D) = \begin{pmatrix} a_1(D)g_{11}(D) \\ a_1(D)g_{12}(D) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ a_2(D)g_{21}(D) \\ a_2(D)g_{22}(D) \end{pmatrix} \tag{7}
\]

\[\Delta = c_1(D) \oplus c_2(D), \tag{8}\]

where \(a(D) = \begin{pmatrix} a_1(D) \\ a_2(D) \end{pmatrix}\) is the 2 bits/sample index sequence and \(a_1(D)\) (\(a_2(D)\)) is the sequence of more (less) significant bit in \(a(D)\), where \(\oplus\) denotes modulo 2 addition. Note that \(c_1(D) = \begin{pmatrix} a_1(D)G_1(D) \\ 0 \end{pmatrix}\) and \(c_2(D) = \begin{pmatrix} 0 \\ a_2(D)G_2(D) \end{pmatrix}\) are the two original codes after a proper shift in bit location. The embedded code is thus the direct sum of these two codes, and can be realized by the tensor product of the original trellises.

Now we investigate the properties of the embedded trellis code constructed as above. In particular, we are concerned about if \(G(D)\) is non-catastrophic and minimal, since these properties determine the trellis complexity and error propagation. The following results are obtained.
Proposition 1 The generator matrix $G(D)$ in (6) is non-catastrophic if and only if the generator matrices of the constituent codes, $G_1(D)$ and $G_2(D)$, are both non-catastrophic, and $\text{GCD}(g_{11}(D), g_{22}(D)) = 1$.

The proof of Proposition 1 is essentially by investigating the greatest common divisor $\text{GCD}(g_{11}(D)g_{21}(D), g_{12}(D)g_{22}(D), g_{11}(D)g_{22}(D))$. In particular, when same codes are used for $G_1(D)$ and $G_2(D)$, we have the following corollary.

Corollary 1.1 When $G_1(D) = G_2(D)$, $G(D)$ is non-catastrophic if and only if $G_1(D)$ (or $G_2(D)$) is non-catastrophic.

Define

$$\text{maxdegree} \triangleq \max_{i,j=1,2} \{\text{deg}(g_{ij}(D))\},$$

where $\text{deg}(\cdot)$ denotes the degree of the polynomial. The following result concerns the minimality of $G(D)$.

Proposition 2 The generator matrix $G(D)$ in (6) is minimal if and only if the generator matrices of the constituent codes, $G_1(D)$ and $G_2(D)$, are both non-catastrophic, and $\text{deg}(g_{11}(D)) = \text{maxdegree}$ or $\text{deg}(g_{22}(D)) = \text{maxdegree}$ or $\text{deg}(g_{12}(D)g_{21}(D)) = \max\{\text{deg}(g_{12}(D)), \text{deg}(g_{21}(D))\}.$

Proposition 2 follows from using the greedy algorithm to look for the minimal encoder equivalent to $G(D)$. In particular, if $G_1(D)$ and $G_2(D)$ are the same, we have

Corollary 2.1 When $G_1(D) = G_2(D)$, $G(D)$ is minimal if and only if $G_1(D)$ (or $G_2(D)$) is non-catastrophic.

3 Proposed algorithm

In this section, we describe and analyze the algorithm for source coding with side information at the decoder using the embedded trellis codes constructed in the previous section. We restrict our attentions to the situation $Y = X + U$, where $U$ is zero-mean Gaussian and independent of $X$.

The proposed algorithm is depicted in Fig. 3. From now on, we leave out the D-transform notation for brevity, and use the boldface letter to represent a sequence. The source sequence $x = (x_1, x_2, \cdots)$ are independent and identically distributed (i.i.d.) according to some known distribution $f(x)$. The encoder uses a trellis-coded quantizer (TCQ) [7] with generator matrix $G(D)$ as in (6). In more details, it searches among all the 2 bits/sample sequences $a$'s for the quantization index sequence $a^*$ such that the distortion measure is minimized:

$$a^* = \arg\min_a d(x, Ga).$$

This search can be carried out by the Viterbi algorithm.
To divide the index sequences $a$'s into bins, we simply assign all the sequences with the same $a_2$, the sequence of less significant bit, into one bin, and label that bin with the index $a_2$. Therefore, after quantization, the bin index $a_2^*$ is sent to the decoder.

The decoder, which receives $a_2^*$, also observes the side information $y = (y_1, y_2, \cdots)$. With the help of $y$, which in fact is a noisy version of $x$, the decoder tries to recover the quantization index $a^*$ in the bin of $a_2^*$. From (14) in the next section, we can see that this is a detection problem in additive white Gaussian noise. The maximum-likelihood detector is given by

$$\hat{a} = \arg\min_{a \in \mathcal{B}(a_2^*)} \|y - Ga\|^2,$$

(11)

where $\mathcal{B}(a_2^*) = \{a \text{ with } a_2 = a_2^*\}$ is the bin of $a_2^*$. By (7)-(9), we can see that (11) can be found by a Viterbi algorithm on the trellis of $G_1(D)$. Finally, the minimum distortion estimation $\hat{x}$ can be found by

$$\hat{x} = \arg\min_{x} E[d(X, x) | X \in \Lambda(\hat{a}), Y = y],$$

(12)

where $\Lambda(\hat{a}) = \{x : \text{the quantization index of } x \text{ is } \hat{a}\}$ is the Voronoi region of $\hat{a}$.

From (12), it can be seen that the overall system performance is determined by the quantization performance at the encoder together with the probability of decoding error $P_e = \text{Prob}(\hat{a} \neq a^*)$. Using these two quantities to measure the performance elucidates the source/channel coding nature of this problem (also see (4)), which has been noticed in [5]. Here, trellis codes are used for both the quantization and the “channel” decoding. In the remaining of this section, we show that the benefits of trellis codes are indeed achieved by the proposed embedded trellis code.

Denote $\nu_t = \max_{J=1,2}\{\deg(g_J(D))\}$. Since the trellis code defined by (6) takes value in the superset $\{D_0, D_2, D_4, D_6\}$ or $\{D_1, D_3, D_5, D_7\}$, it can be shown that

**Theorem 1** If $G(D)$ in (6) is minimal, then the trellis-coded quantizer using $G(D)$ can achieve the granular gain (ensemble performance) of TCQ of $2^{\nu_t+1}$ states.

Furthermore, denote by $C_1$ ($C_2$) the set of codeword sequences $c_1(D)$ ($c_2(D)$) defined in (9). We have

**Proposition 3** $\|c_1 - c_1'\| = \|(c_1 \oplus c_2) - (c_1' \oplus c_2)\|$ for any $c_1, c_1' \in C_1$ and any $c_2 \in C_2$.

Proposition 3 shows that the Euclidean distance between any two codewords in code $C_1$ is unchanged by adding a codeword in code $C_2$ to them. That is, for the $C_1$
code, adding a codeword of $C_2$ is an isometry. The proof is basically by checking all the cases in one dimension. Since the distance measure is additive, the proposition follows. Based on this result, it can be seen that each subcode, formed by the codewords in each bin, is isomorphic to the code $C_1$. Therefore, all the subcodes have the same performance of that of code $C_1$. Hence, we have shown that

**Theorem 2** With $G(D)$ as in (6) and algorithm as described, the coding gain of $G_1(D)$ is achieved at the decoder.

4 Simulation results and performance analysis

We use Monte Carlo simulations for the Gaussian source $X$ to verify the performance of the proposed algorithm. Trellis codes in [7] are used as the constituent codes and we choose $G_1(D) = G_2(D)$. The quantization performance (granular gain in dB) is shown in Table 1, compared with the scalar quantizer and the rate-distortion limit.

<table>
<thead>
<tr>
<th>$\nu_1 = \nu_2 =$</th>
<th>scalar quantizer</th>
<th>$D(R)$ limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.16</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1.26</td>
<td>1.33</td>
</tr>
<tr>
<td>4</td>
<td>1.33</td>
<td>1.53</td>
</tr>
</tbody>
</table>

Table 1: Granular gain.

Fig. 4 shows the simulation result for the probability of decoding error when using a 2-bit quantizer ($\nu_1 = \nu_2 = 2$) and transmitting only 1 bit/sample. It is also compared with the result in [5], and about 1 dB more coding gain is achieved by the proposed algorithm at $P_e = 10^{-6}$. The improvement over [5] is due to the fact that the Voronoi regions of TCQ are more sphere-like.

![Figure 4: Probability of decoding error.](image-url)
In [6], simulation results for three lattices, namely $A_2$, $E_8$ and $A_{24}$, are obtained, and it is shown that $E_8$ achieves the best performance among the three lattices. Since the packing performance of a simple 4-state trellis is better than that of the $E_8$ lattice [7, 9], it can be shown that the proposed algorithm also outperforms the results in [6], in addition to the complexity advantage of trellis codes over lattice codes.

The effect of the transmission rate $\tilde{R}$ on the performance of decoding can be analyzed as follows. We model the quantization noise $U_q$ as zero-mean, white Gaussian with variance $\sigma_q^2$, independent of the quantization output (codeword) $X_q$, and independent of $U$. While this model is only exact for optimal theoretical quantizers, it can be made a more accurate model for the practical quantizers by using dithered quantization [10], provided that a common random sequence is available at both the encoder and the decoder. Now,

$$X = X_q + U_q,$$

and

$$Y = X + U = X_q + U_q + U.$$

(13)

(14)

Note that $(U_q + U)$ is the noise in effect for detection at the decoder, which is additive white Gaussian and has variance

$$\sigma^2 = \sigma_q^2 + \sigma_U^2,$$

(15)

$$\approx (2^{-2R}\frac{\sigma_q^2}{\sigma_U^2} + 1)\sigma_U^2,$$

(16)

$$\approx (2^{-2(R-C)} + 1)\sigma_U^2,$$

(17)

$$= (2^{-2R} + 1)\sigma_U^2,$$

(18)

where (16) assumes optimal quantization and (17) assumes high SNR ($\sigma_q^2 \gg \sigma_U^2$). Therefore, the effect of the transmission rate $\tilde{R}$ boils down to a loss of coding gain (in dB) by

$$10 \log_{10}(1 + 2^{-2R}) \approx 10 \log_{10}(e) + 2^{-2R}$$

(19)

when $\tilde{R}$ is large. Note that this amount decreases exponentially in $\tilde{R}$.

Fig. 5 shows the simulation results of the effect of $\tilde{R}$ on the decoding performance. It can be seen that they agree with the analysis given in (19). When $\tilde{R} > 2$, little can be gained by increasing $\tilde{R}$.

From Fig. 5, it can also be seen that there is about 4.5 dB gap between the high rate performance and the theoretical limit, which is exactly the performance of TCM with $\nu = 2$ [8]. This is due to the fact that at high rates, the disconnected Voronoi regions in each bin become very small, and thus $\sigma^2 \rightarrow \sigma_U^2$ (regardless of the model of $U_q$). Then, by Theorem 2, the TCM performance is achieved.

Therefore, the high rate assumption, in essence, disentangles the entire problem into a pure quantization problem and a pure modulation problem. Trellis codes can be designed independently for these two problems, and the rate-distortion function as well as the channel capacity can both be approached with increasing trellis complexity. Then, by unifying the trellis codes into our embedded trellis structure, the information-theoretic limit (c.f.(4)) can be approached at high rates.
5 Summary

We construct an embedded trellis code structure, such that in the sequence space, its signal constellations at different levels are embedded. The complexity and error propagation issues of this structure are studied through investigating the properties of minimality and non-catastrophe of the encoder matrix. It is also shown that the signal constellations of the constructed code have good packing property at both levels if the constituent codes are good.

When applied to the problem of source coding with side information at the decoder, the proposed embedded trellis code gives a 1.16 dB granular gain at the encoder and about 1 dB more coding gain at the decoder compared with the scheme in [5], with simple 4-state constituent trellis codes. It is also shown that, with high rate of $R$, the proposed algorithm can approach the information-theoretic limit when the trellis complexity increases.

The embedded trellis code structure constructed in this paper is very generic, and can probably be applied to many other applications, such as modulation, joint source/channel coding, et al. They are up to future investigation.

Acknowledgement

The authors are grateful to S. S. Pradhan and K. Ramchandran for the interesting discussions which introduced them to this problem. They also would like to thank the anonymous reviewers for the helpful comments and for pointing out the work on lattice codes in [6].
References


